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Low-complexity tracking and estimation of frequency and amplitude of sinusoids

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Abstract

Formulas are proposed for estimating and tracking the (time-dependent) frequency and amplitude of a sinusoidal signal. These formulas are recursive in nature, and use only the instantaneous values of the signal, in a low cost and low complexity manner; in particular, there is no need to take square roots or to carry out divisions. Special attention is paid to the convergence behavior of the algorithm for stationary signals and the dynamic behavior in case of a transition to another stationary state. The latter issue is considered to be important to assess the tracking abilities for non-stationary signals.

Keywords: Frequency tracking; Frequency estimation; Real-time tracking algorithm

1. Introduction

In many engineering applications, such as in astronomy, acoustics, and communication, one likes to know the frequency of a signal. Other applications like (acoustic) velocity measurements of Doppler shift, frequency shift keying (FSK), sonar, and biometrical, see [1] for more applications. Recently an algorithm was devised for rapid power-line frequency monitoring [2], based on a number of formulas presented in [3]. There is a vast amount of literature on this topic including a text book [4]. A comparative study of four adaptive frequency trackers is given in [5]. While most of the algorithms presented in [4] are advanced and not very suitable to implement in real-time, the algorithm proposed in the present paper is striving for maximum efficiency, by avoiding division, trigonometric operations such as FFT—which also necessitates the use of buffers—and the like. Another difference is that the advanced methods are meant for tracking in bad SNRs, so their range

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of applications is different from what here is envisaged. We pay special attention to the convergence behavior of the algorithm for stationary signals and the dynamic behavior in case of a transition to another stationary state. The latter issue is considered important for assessing the tracking abilities for non-stationary signals.

We shall show in Section 2 that the recursion

$$\hat{r}_k = \hat{r}_{k-1} + x_{k-1}\gamma[x_k + x_{k-2} - 2x_{k-1}\hat{r}_{k-1}] \tag{1}$$

can be used to get a good approximation of the frequency of a signal given by

$$r_k = \cos\bigl(\omega_0(k)T_s\bigr), \tag{2}$$

where $\omega_0(k)$ is the frequency of the input signal $x(k)$ to be determined, k is the time index, and $f_s = 1/T_s$ is the sampling frequency. The 'hat' on $\hat{r}_k$ denotes that it is an approximated value of r_k. The parameter γ determines the convergence speed, and hence determines the tracking behavior of $\hat{r}$, but not the actual value of $\lim_{k\to\infty} \hat{r}_k$ in the stationary case.

Equation (1) is the basis for our approach of recursively tracking the frequency.

In Section 3 we shall analyze the solution of Eq. (1), starting from an initial value $\hat{r}_0$ at $k = 0$, when $\gamma \downarrow 0$, and we shall indicate conditions under which

$$\lim_{\gamma\downarrow 0}\Bigl[\lim_{k\to\infty} \hat{r}_k\Bigr] = \cos(\omega_0 T_s). \tag{3}$$

While the analysis is similar to that of an other algorithm [6], the present papers differs in two aspects. First, the former method tracks the correlation coefficient of two signals, while the present method tracks the frequency of a single signal. A second difference is that the former method considers in the final tracking formula signals with equal RMS values while the present method does not have this restriction. The final tracking formula differ in both methods. The analysis in [6] was facilitated considerably by switching from (discrete-time) difference equations as in Eq. (1) to (continuous-time) differential equations, and the same approach shall be followed here.

In Section 4 we consider the case of a sinusoidal input signal x, and we compute explicitly the left-hand side of Eq. (3) for the solution of Eq. (1). It turns out that the recursion Eq. (1) yields the correct value r for the left-hand side of Eq. (3). At the end of Section 4 we derive Eq. (34) to track the (squared) amplitude of the signal x.

Finally, conclusions are given in Section 5.

2. Derivation of tracking formulas

In this section we consider r as defined in Eq. (2), and we show that r satisfies to a good approximation (when γ is small) the recursion in Eq. (1).

We start with Adelson's equation from [2]

$$r = \frac{\sum_{j=1}^{n-1} x_j(x_{j-1} + x_{j+1})}{2\sum_{j=1}^{n-1} x_j^2}. \tag{4}$$

In order to make this formula suitable for tracking purposes, it is modified into

$$r_k = \frac{\sum_{j=1}^{n-1} x_{k-j}(x_{k-j-1} + x_{k-j+1})}{2\sum_{j=1}^{n-1} x_{k-j}^2}. \tag{5}$$

Now r_k depends on $n-1$ samples from the past and the current sample x_k. However, it is not optimal for tracking purposes, since it suffers from the fact that it requires many operations and may lead to numerical difficulties in the case of a small denominator in Eq. (5). Therefore, a second modification is made by using—instead of a rectangular window and an averaging over $2n$ $x_i x_{i+1}$ products—an exponential window. In order to minimize the number of operations we select $n = 2$ in Eq. (5). Now we define (indices n for numerator, and d for denominator)

$$r(k) = \frac{S_n}{S_d}, \tag{6}$$

where

$$S_n(k) = \sum_{l=0}^{\infty} c\, e^{-\eta l} x_{k-l-1}(x_{k-l} + x_{k-l-2}), \tag{7}$$

$$S_d(k) = \sum_{l=0}^{\infty} 2c\, e^{-\eta l} x_{k-l-1}^2, \tag{8}$$

$$c = 1 - e^{-\eta}, \tag{9}$$

and η is a small but positive number that should be adjusted to the particular circumstances for which tracking of the frequency is required. We now start to show that r of Eqs. (6)–(9) satisfies to a good approximation the recursion in Eq. (1). To this end we note that

$$S_n(k) = e^{-\eta} S_n(k-1) + c x_{k-1}(x_k + x_{k-2}), \tag{10}$$

and

$$S_d(k) = e^{-\eta} S_d(k-1) + 2c x_{k-1}^2. \tag{11}$$

Hence, from the definition in Eq. (6),

$$r(k) = \frac{S_n(k-1) + c\, e^{\eta} x_{k-1}(x_k + x_{k-2})}{S_d(k-1) + 2c\, e^{\eta} x_{k-1}^2}. \tag{12}$$

Since we consider small values of η we have that $c = 1 - e^{-\eta}$ is small as well. Expanding the right-hand side of Eq. (12) in powers of c and retaining only the constant and the linear term, we get after some calculations

$$r(k) = r(k-1) + \frac{c\, e^{\eta}}{S_d(k-1)} x_{k-1}\big[x_k + x_{k-2} - 2r(k-1)x_{k-1}\big] + O(c^2). \tag{13}$$

Then, deleting the $O(c^2)$ term, we obtain the recursion in Eq. (1) when we identify

$$x_{\text{RMS}}^2 = S_d(k), \tag{14}$$

for a sufficiently large k, and

$$\gamma = \frac{c\,e^{\eta}}{x_{\mathrm{RMS}}^2}, \tag{15}$$

which is a constant for a stationary signal x.

We observe at this point that we have obtained the recursion in Eq. (1) by applying certain approximations (as in Eq. (14)) and neglecting higher order terms. Therefore, it is not immediately obvious that the actual r of Eq. (2) and the solution of $\hat{r}$ of the recursion in Eq. (1) have the same value, in particular for large k. In Section 3, however, we shall show that $\hat{r}$ and r are closely related for the purposes of frequency estimation.

3. Analysis of the solution of the basic recursion

In this section we consider the basic recursion in Eq. (1), and we analyze its solution $\hat{r}(k)$, given an initial value $\hat{r}_0$ at $k = 0$, when $\gamma \downarrow 0$. We do this by reformulating the recursion in Eq. (1) so that it assumes the same form as the recursion in [6]. It appears to be convenient to introduce the new variables

$$\beta_k = 2x_{k-1}^2, \tag{16}$$

and

$$\delta_k = x_{k-1}(x_k + x_{k-2}). \tag{17}$$

Thus, we shall consider the recursion in Eq. (1) which we rewrite as

$$\hat{r}(k) = (1 - \gamma\beta_k)\hat{r}(k-1) + \gamma\delta_k \tag{18}$$

for $k = 1, 2, \ldots$, with γ a small positive parameter and δ_k, β_k bounded sequences with $0 \leqslant \beta_k \leqslant 1$.

In [6] it was shown how to obtain the limiting behavior of $\hat{r}(k)$ as $k \to \infty$ when $\gamma > 0$ is small. This was done under an assumption (slightly stronger than required) that the mean values (denoted by $M[.]$)

$$b_0(\gamma) = M\left[\frac{-1}{\gamma}\log(1 - \gamma\beta_k)\right] = \lim_{K\to\infty}\frac{1}{K}\sum_{l=1}^{K}\frac{-1}{\gamma}\log(1 - \gamma\beta_l),$$

$$d_0 = M[\delta_k] = \lim_{K\to\infty}\frac{1}{K}\sum_{l=1}^{K}\delta_l \tag{19}$$

for the discrete-time case and

$$b_0 = M\big[\beta(t)\big] = \lim_{T\to\infty}\frac{1}{T}\int_0^T \beta(s)\,ds,$$

$$d_0 = M\big[\delta(t)\big] = \lim_{T\to\infty}\frac{1}{T}\int_0^T \delta(s)\,ds, \tag{20}$$

for the corresponding continuous-time case, exist.

Since $b_0(\gamma) \to b_0$ as $\gamma \downarrow 0$ it was shown [6] that

$$\lim_{\gamma\downarrow 0}\Big[\lim_{k\to\infty} \hat{r}(k)\Big] = \frac{M[\delta_k]}{M[\beta_k]} = \frac{d_0}{b_0} = \frac{M[\delta(t)]}{M[\beta(t)]} \tag{21}$$

and, for any number $b < b_0(\gamma)$, that

$$\hat{r}(k) = \frac{d_0}{b_0(\gamma T_s)} + O(e^{-\gamma b k T_s}), \quad k \geqslant 0. \tag{22}$$

This shows that the time constant, i.e. the time for the exponential term to drop to e^{-1} of its original value, for the tracking behavior is given by

$$\tau = \frac{T_s}{\gamma b_0(\gamma T_s)}. \tag{23}$$

We finally observe that, $b_0(\gamma) \to b_0$ as $\gamma \downarrow 0$. In Section 4 we shall work this out for sinusoidal signals x.

4. Sinusoidal input signals

In this section we test the algorithm derived in Section 2, and analyzed in Section 3 with respect to the steady state behavior, for sinusoidal input signals. Hence we take

$$x_k = A_0 \sin(\omega_0 k T_s + \phi), \tag{24}$$

with arbitrary A_0 and ϕ. Calculating β and δ with Eqs. (16)–(17), and using Eq. (21) it is easy to obtain that

$$\lim_{\gamma\downarrow 0}\Big[\lim_{k\to\infty} \hat{r}(k)\Big] = \cos\omega_0 T_s, \tag{25}$$

compare with Eq. (2), and this limit obviously does not depend on A_0 nor on ϕ. If Eq. (24) and $r_{k-1} = \cos\omega_0 T_s$ is substituted into Eq. (1) then we get $r_k = r_{k-1}$, independent on γ, indicating that r remains on a constant converged value. Using Eq. (23) and $b_0 = A_0^2$ it appears that the time constant of the tracking behavior is equal to

$$\tau_d = T_s \big/ \big(\gamma A_0^2\big). \tag{26}$$

Consider the case that the signal x consists of two sinusoids (with unequal frequencies), where the latter can represent a disturbance, such as a harmonic distortion, of the first sinusoid. Thus

$$x_k = A_0 \sin(\omega_0 k) + A_1 \sin(\omega_1 k), \tag{27}$$

and by using Eqs. (16)–(17), and Eq. (21) we get

$$\lim_{\gamma\downarrow 0}\Big[\lim_{k\to\infty} \hat{r}(k)\Big] = \frac{A_0^2\cos\omega_0 T_s + A_1^2\cos\omega_1 T_s}{A_0^2 + A_1^2}. \tag{28}$$

Here we see that adding a second sinusoid, the result (Eq. (28)) deviates from that of a single one (Eq. (25)), but as long as $A_1 \ll A_0$, this will be a small effect only. Consider the

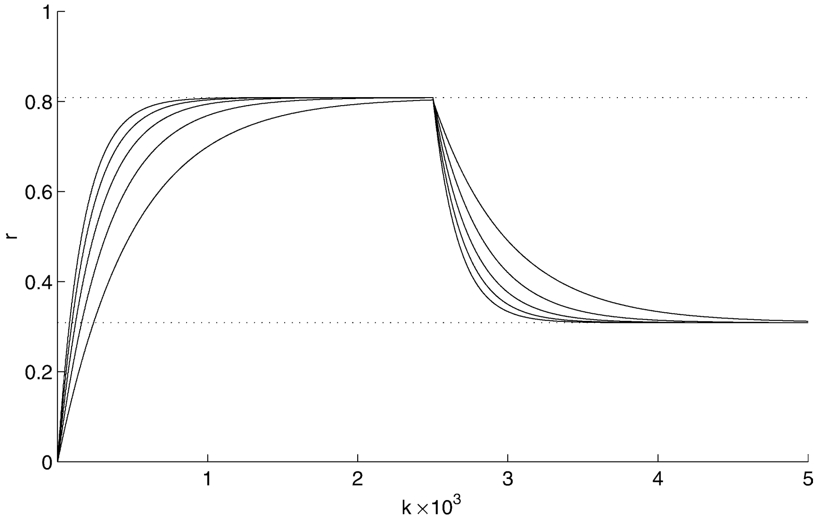

Fig. 1. Step response on r using Eq. (1) vs normalized time (or sample index) k for a sinusoidal input signal with amplitude $A_0 = 1$, $\hat{r}_0 = 0$, and γ as parameter, and making a step from $\omega_0 T_s = \pi/5$ ($r = 0.81$) to $\omega_0 T_s = 2\pi/5$ ($r = 0.31$). The dotted lines are the final values given by Eq. (25). The parameter γ is changing from 2×10^{-3} to 6×10^{-3} with steps of 1×10^{-3}, and is causing steeper edges in r for larger values of γ.

case that the signal x consists of a sinusoid with additional noise $n(k)$ (with autocorrelation function $R_n(k)$). Thus

$$x_k = A_0 \sin(\omega_0 k) + n(k), \tag{29}$$

and we get

$$\lim_{\gamma \downarrow 0} \Big[\lim_{k \to \infty} \hat{r}(k) \Big] = \frac{{A_0}^2 \cos \omega_0 T_s + 2 R_n(T_s)}{{A_0}^2 + 2 R_n(0)}. \tag{30}$$

Here we see that adding a noise to the sinusoid, the result (Eq. (30)) deviates from that of a single one (Eq. (25)), but as long as $n(k) \ll A_0$, this will be a small effect only. Equation (30), shows that if $R_n(T_s)$ and $R_n(0)$ are known or can be estimated, the estimate of $\hat{r}$ can be easily improved.

To demonstrate the tracking behavior of Eq. (1), in Fig. 1 the step response is plotted for a sinusoidal input signal, making a change in frequency, for various values of γ. It appears that the time constants correspond well with the values predicted by Eq. (26). The values of γ used in Fig. 1 are just for illustration purposes, but may be much larger. To obtain stability we need $|1 - \beta_k \gamma| < 1$. Practical values for sinusoidal input signals are $0 < A_0 \gamma < 0.5$.

The same procedure as for tracking the frequency, can be followed in order to track the amplitude A_0 of the input signal. By simple trigonometry we see that for a sinusoidal signal x as Eq. (24) we get the identity

$$x_{k-1}^2 - x_{k-2} x_k = A_0^2 \sin^2(\omega_0 T). \tag{31}$$

Now using Eq. (25) and Eq. (31) A_0^2 can be tracked. To that end β and δ in Eq. (16) and Eq. (17) are modified into

$$\beta_k' = 1 - r_k^2, \tag{32}$$

and

$$\delta_k' = x_{k-1}^2 - x_k x_{k-2}. \tag{33}$$

Using Eq. (18) we get

$$A(k) = (1 - \gamma\beta_k')A(k-1) + \gamma\delta_k', \tag{34}$$

where $\hat{A}_0 = \sqrt{A(k)}$.

5. Conclusions

This paper has presented formulas for tracking the frequency and amplitude of a sinusoidal signal in real-time. The proposed method aims at lowering the computational complexity compared to the other methods. It has been shown that the proposed method contains only a few arithmetic operations, and is insensitive to the initial value. The behavior of the algorithm has been shown to provide satisfactory accuracy for sinusoidal inputs.

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