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# Estimating the velocity profile and acoustical quantities of a harmonically vibrating loudspeaker membrane from on-axis pressure data

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## ABSTRACT

Formulas are presented for acoustical quantities of a harmonically excited resilient, flat, circular loudspeaker in an infinite baffle. These quantities are the sound pressure on-axis, far-field, directivity, and the total radiated power. These quantities are obtained by expanding the velocity distribution in terms of orthogonal polynomials. For rigid and non-rigid radiators, this yields explicit, series expressions for both the on-axis and far-field pressure. In the reverse direction, a method of estimating velocity distributions from (measured) on-axis pressures by matching in terms of expansion coefficients is described. Together with the forward far-field computation scheme, this yields a method for assessment of loudspeakers in the far-field and of the total radiated power from (relatively near-field) on-axis data (generalized Keele scheme).

## 1. INTRODUCTION

In this paper an analytic method developed in [1, 2] for the calculation of acoustical quantities such as the sound pressure on-axis, far-field, directivity, and the total radiated power is presented to the audio engineering community. This method is based on the

analytical results as developed in the diffraction theory of optical aberrations by Nijboer [3] and Zernike and Nijboer [4], see also [5, 6]. Using this approach, many of the analytic results in Greenspan [7], such as those on the sound pressure on-axis, the total radiated power, and the results in text books [8] on

far-field expressions and directivity can be presented and extended in a systematic fashion. This is worked out in [1] for the results on on-axis pressure and far-field expressions for arbitrary velocity distributions on flat piston radiators. The mathematical foundation of these methods related to directivity, and the total radiated power is discussed in [2]. An arbitrary velocity distribution can be efficiently developed as a series in Zernike polynomials. Using near-field pressure measurements on-axis the coefficients of these polynomials can be estimated. With these estimated coefficients the acoustical quantities mentioned above can be estimated as well. An immediate application is to predict far-field sound pressure data from near-field pressure data measured without using an anechoic chamber (generalized Keele scheme [1, 9]).

The radiated pressure is given in integral form by the Rayleigh integral [8, 10] as

$$p(\underline{r}, t) = \frac{i\rho_0 c k}{2\pi} e^{i\omega t} \int_S v(\underline{r}_s) \frac{e^{-ikr'}}{r'} dS, \quad (1)$$

where  $\rho_0$  is the density of the medium,  $c$  is the speed of sound in the medium,  $k = \omega/c$  is the wave number and  $\omega$  is the radian frequency of the harmonically vibrating surface  $S$ . Furthermore,  $t$  is time,  $\underline{r}$  is a field point,  $\underline{r}_s$  is a point on the surface  $S$ ,  $r' = |\underline{r} - \underline{r}_s|$  is the distance between  $\underline{r}$  and  $\underline{r}_s$ , and  $v(\underline{r}_s)$  is the normal component of a (not necessarily uniform) velocity profile on the surface  $S$ . The time variable  $t$  in  $p(\underline{r}, t)$  and the harmonic factor  $\exp(i\omega t)$  in front of the integral in Eq. (1) will be omitted in the sequel. For transparency of exposition, the surface  $S$  is assumed [1, 2] to be a disk of radius  $a$ ,  $|\underline{r}_s| = r_s \leq a$ , with average velocity  $V_s$ ; in [1] a generalization to the case of dome-shaped radiator surfaces  $S$  is done. See Fig. 1 for the geometry and notations used in the case of a flat piston. The volume velocity  $V$  at the piston is

$$V = \int_S v(\underline{r}_s) dS = V_s \pi a^2. \quad (2)$$

Frankort [11] has shown that loudspeaker cones mainly vibrate in a radially symmetric fashion. Therefore the attention in this paper is restricted to radially symmetric velocity distributions  $v$ , which

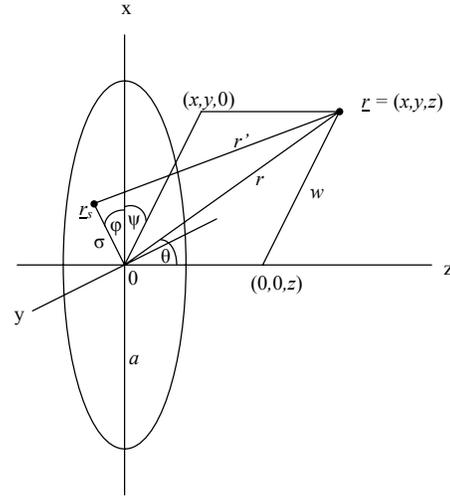


Fig. 1: Set-up and notations.

$$\begin{aligned} \underline{r}_s &= (x_s, y_s, 0) = (\sigma \cos \varphi, \sigma \sin \varphi, 0) \\ \underline{r} &= (x, y, z) = (r \sin \theta \cos \psi, r \sin \theta \sin \psi, r \cos \theta) \\ w &= r \sin \theta = (x^2 + y^2)^{1/2}, \quad z = r \cos \theta \\ r &= |\underline{r}| = (x^2 + y^2 + z^2)^{1/2} = (w^2 + z^2)^{1/2} \\ r' &= |\underline{r} - \underline{r}_s| = (r^2 + \sigma^2 - 2\sigma w \cos(\psi - \varphi))^{1/2}. \end{aligned}$$

are denoted as  $v(\sigma)$ ,  $0 \leq \sigma \leq a$ . Under an integrability condition, these  $v$ 's admit a representation

$$v(\sigma) = V_s \sum_{n=0}^{\infty} u_n R_{2n}^0(\sigma/a), \quad 0 \leq \sigma \leq a, \quad (3)$$

in which the  $u_n$  are scalar coefficients and

$$R_{2n}^0(\rho) = P_n(2\rho^2 - 1), \quad 0 \leq \rho \leq 1, \quad (4)$$

where  $P_n$  are the Legendre polynomials [12]. In [1] there are presented analytical results for the on-axis and far-field pressure  $p(\underline{x})$  in Eq. (1) related to the coefficients  $u_n$  and polynomials  $R_{2n}^0$  occurring in the expansion in Eq. (3).

By orthogonality of the terms  $R_{2n}^0(\rho)$ , the coefficients  $u_n$  in Eq. (3) can be found in integral form as

$$u_n = \frac{2(2n+1)}{V_s} \int_0^1 R_{2n}^0(\rho) v(a\rho) \rho d\rho, \quad n = 0, 1, \dots \quad (5)$$

In particular,  $u_0 = 1$ . There is an impressive number of cases where one can explicitly find the  $u_n$  in

Eq. (5); these include the rigid piston ( $\ell = 0$ ), the simply supported radiator ( $\ell = 1$ ) and the higher order clamped radiators ( $\ell \geq 2$ ) with the velocity profile given by

$$v^{(\ell)}(\sigma) = (\ell + 1)V_s(1 - (\sigma/a)^2)^\ell H(a - \sigma), \quad \ell = 0, 1, \dots, \quad (6)$$

and the Gaussian velocity profile

$$v(\sigma; \alpha) = \frac{\alpha V_s}{1 - e^{-\alpha}} e^{-\alpha(\sigma/a)^2} H(a - \sigma), \quad (7)$$

where  $H(x)$  is the Heaviside function,  $H(x) = 0, 1/2$ , or  $1$  according as  $x$  is negative, zero, or positive. Hence, the velocity profiles in Eqs. (6) and (7) vanish for  $\sigma > a$ .

The relevance of the Zernike terms  $R_{2n}^0$  for the purposes of the present paper is the existence of closed-form formulas. For instance, the radiators in Eq. (6) give rise to an on-axis pressure expansion in the form of a series of  $n + 1$  terms  $u_\ell j_\ell(kr_-)h_\ell^{(2)}(kr_+)$ , with  $r_\pm$  argument values directly related to the axial position  $(0, 0, r)$ , while the far-field pressure expansion is a similar series involving terms  $u_\ell J_{2\ell+1}(ka \sin \theta)/(ka \sin \theta)$ . In [2] it is shown how the acoustic power  $P$  and the directivity  $D$  are computed from the coefficients  $u_\ell$ . In the reverse direction, the forward computation schemes for the on-axis and far-field pressures is complemented in [1] by an inverse method with potential use in far-field loudspeaker assessment. Here one estimates the expansion coefficients  $u_n$  of a velocity profile  $v$  by matching with a measured on-axis pressure data set and then one predicts the far-field sound radiation using the far-field forward formula.

## 2. PAPER OUTLINE

In Sec. 3 the definition and basic properties of the Zernike polynomials are given, and some of the expansion results that are relevant for this paper are presented. Furthermore, the Hankel transform of the Zernike polynomials is presented in closed form. The latter result is of importance both for the forward computation scheme for the far-field and for establishing results on the radiated power, etc.

In Sec. 4 the basic formulas are highlighted and discussed. Thus, the closed form involving a spherical Bessel and Hankel function for the on-axis pressure

associated with a single Zernike term is presented, with comments on both near-field and far-field behavior and on behavior for low and high frequencies in terms of  $ka$ . Also the far-field expression is presented as derived in [1].

In Sec. 5, the formulas [2] for the total radiated power and directivity are presented and discussed.

In Sec. 6 the inverse method [1] of estimating the Zernike expansion coefficients of the velocity profile from the on-axis (measured, sampled) pressure data is considered. Together with the forward scheme for computing far-field pressures from Zernike expansions, this yields a loudspeaker assessment method that generalizes a well-known method in audio engineering for estimating the far-field of a loudspeaker from near-field on-axis data in the case of a rigid piston (Keele scheme [9]).

In the present paper only a few measurements and simulation results will be shown.

## 3. THE ZERNIKE TERMS $R_{2n}^0$

The Zernike terms  $R_{2n}^0$  are polynomials of degree  $2n$  given by

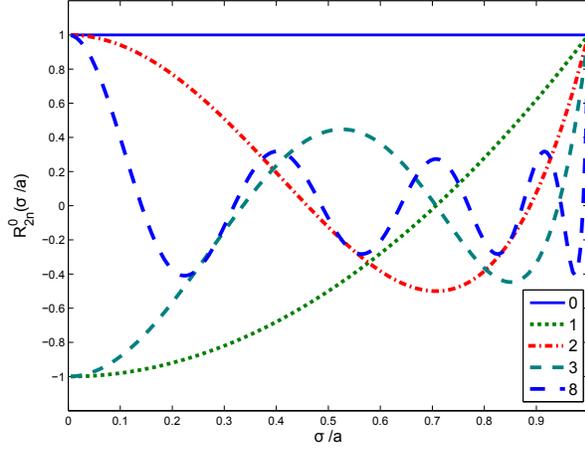
$$R_{2n}^0(\sigma/a) = P_n(2(\sigma/a)^2 - 1) = \sum_{s=0}^n (-1)^s \binom{2n-s}{n} \binom{n}{s} (\sigma/a)^{2n-2s}, \quad (8)$$

where  $P_n$  is the Legendre polynomial of degree  $n$ , see Ref. [12, 22.3.8 and 22.5.42]. The first few  $R_{2n}^0$  are given in Table 1, and in Fig. 2 some of them are plotted as a function of  $\rho = \sigma/a \in [0, 1]$ . The  $R_{2n}^0$  cannot be interpreted directly in physical terms, unlike the velocity profiles  $v^{(\ell)}$  in Eq. (6) in which  $\ell$  has the interpretation of a smoothness parameter for the transition from the non-zero values on the piston ( $\sigma < a$ ) to the value zero outside the piston ( $\sigma > a$ ). Rather, their significance for loudspeaker analysis stems from the following facts.

- They are very efficient and convenient in representing a general velocity profile  $v$ . This is due to the orthogonality property

$$\int_0^1 R_{2n_1}^0(\rho) R_{2n_2}^0(\rho) \rho d\rho = \frac{\delta_{n_1 n_2}}{2(2n_1 + 1)} \quad (9)$$

(where  $\delta$  is Kronecker's delta), as well as the fact that many velocity profiles considered in loudspeaker analysis can be represented as a Zernike



**Fig. 2:** The Zernike terms  $R_{2n}^0$  vs.  $\sigma/a$  for various values of  $n$ .

series. In [1][Appendix A], a number of cases are listed, such as the expansion

$$(1 - (\sigma/a)^2)^\ell = \sum_{n=0}^{\ell} (-1)^n \frac{2n+1}{n+1} \binom{\ell}{n} R_{2n}^0(\sigma/a), \quad (10)$$

which are relevant for the rigid and simply supported ( $\ell = 0, 1$ ) and the clamped radiators ( $\ell \geq 2$ ) in Eq. (6), and the expansion

$$e^{-\alpha\sigma^2} = e^{-\frac{1}{2}\alpha} \sum_{n=0}^{\infty} (-1)^n (2n+1) \frac{\sqrt{\pi}}{\alpha} I_{n+1/2}(\alpha/2) R_{2n}^0(\sigma/a), \quad (11)$$

which is relevant for the truncated Gaussian radiator in Eq. (7).

- The Hankel transform of 0<sup>th</sup> order of the  $R_{2n}^0$  has a closed form, viz.

$$\int_0^a J_0(u\sigma) R_{2n}^0(\sigma) \sigma d\sigma = (-1)^n \frac{a}{u} J_{2n+1}(ua). \quad (12)$$

This formula has been proved in Ref. [3] as a special case of a formula expressing the  $m^{\text{th}}$  order Hankel transform of Zernike polynomials of azimuthal order  $m$  in terms of Bessel functions of the first kind. This formula is very important for the development of explicit analytic results in the spirit of [7]. It gives for instance, the far-field expression for the pressure due to a single

term  $R_{2n}^0$  in the velocity profile, see Sec. 4.2 and [1][Appendix B].

**Table 1:** Zernike polynomials

$n$	$R_{2n}^0(\sigma/a)$
0	1
1	$2(\sigma/a)^2 - 1$
2	$6(\sigma/a)^4 - 6(\sigma/a)^2 + 1$
3	$20(\sigma/a)^6 - 30(\sigma/a)^4 + 12(\sigma/a)^2 - 1$

#### 4. ON-AXIS AND FAR-FIELD EXPRESSIONS

The velocity profile  $v(\sigma)$  considered in this section (normal component) vanishes outside the disk  $\sigma \leq a$  and has been developed into a Zernike series as in Eq. (3) with coefficients  $u_n$  given in accordance with Eq. (5) or explicitly as in the cases discussed earlier.

##### 4.1. On-axis expression

There holds [1] for an on-axis point  $\underline{r} = (0, 0, r)$  with  $r \geq 0$  the formula

$$p(\underline{r}) = \frac{1}{2} \rho_0 c V_s (ka)^2 \sum_{n=0}^{\infty} \gamma_n(k, r) u_n, \quad (13)$$

in which

$$\gamma_n(k, r) = (-1)^n j_n(kr_-) h_n^{(2)}(kr_+), \quad (14)$$

$$r_{\pm} = \frac{1}{2} (\sqrt{r^2 + a^2} \pm r).$$

The  $r_{\pm}$  in Eq. (14) satisfy

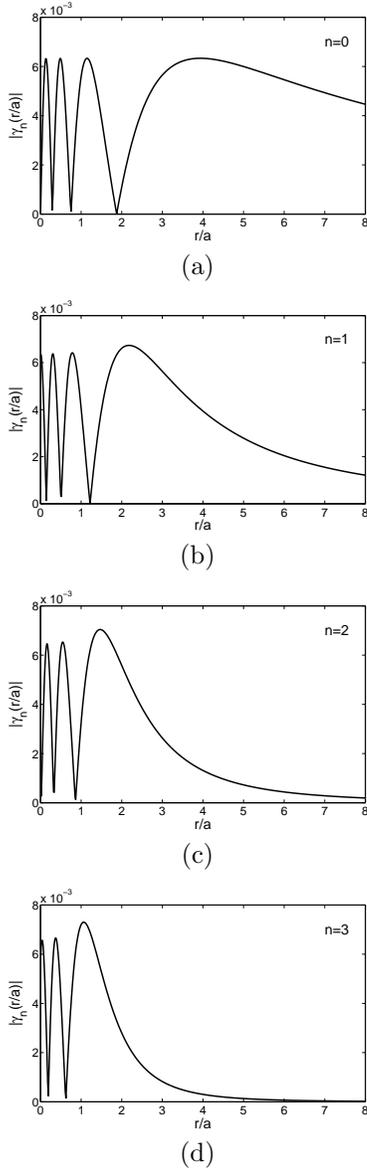
$$0 \leq r_- \leq \frac{1}{2}a \leq r_+, \quad r_+ r_- = \frac{1}{4}a^2, \quad (15)$$

$$r_+ + r_- = \sqrt{r^2 + a^2}.$$

The  $j_n$  and  $h_n^{(2)} = j_n - i y_n$  are the spherical Bessel and Hankel function, respectively, of the order  $n = 0, 1, \dots$ , see Ref. [12, § 10.1.]. In particular,  $j_0(z) = (\sin z)/z$  and  $h_0^{(2)}(z) = (ie^{-iz})/z$ .

Figure 3 shows a plot of  $|\gamma_{\ell=n=0}(k, r)|$  as a function of  $r/a$  (rigid piston) and of  $|\gamma_n(k, r)|$  for  $n = 1, 2, 3$ . Some comments on these plots are presented at the end of this subsection.

The result in Eqs. (13) and (14) comprises the known result [8, 8.31a,b] for the rigid piston ( $\ell = 0$  in



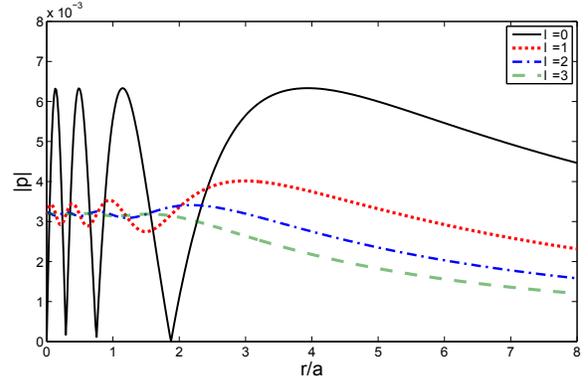
**Fig. 3:** The product  $|j_n(kr_-)h_n^{(2)}(kr_+)|$  from Eq. (14), for  $n = 0, \dots, 3$ , vs.  $r/a$ , where  $a/\lambda = 4$ , and  $a = 0.1$  m, which yields  $f = 13.7$  kHz and  $ka = 8\pi$ . (a)  $n = 0$ , (b)  $n = 1$ , (c)  $n = 2$ , (d)  $n = 3$ .

Eq. (10) and  $u_0 = 1$ ,  $u_1 = u_2 = \dots = 0$ ) with  $p(\underline{r})$ ,  $\underline{r} = (0, 0, r)$ , given by

$$\begin{aligned} p(\underline{r}) &= \frac{1}{2}\rho_0 c V_s (ka)^2 \frac{\sin kr_-}{kr_-} \frac{ie^{-ikr_+}}{kr_+} \\ &= 2i\rho_0 c V_s e^{-\frac{1}{2}ik((r^2+a^2)^{\frac{1}{2}}+r)} \cdot \\ &\quad \sin \frac{1}{2}k((r^2+a^2)^{\frac{1}{2}}-r), \end{aligned} \quad (16)$$

and it generalizes immediately to the case of the simply supported radiator ( $\ell = 1$ ) and the clamped radiators ( $\ell \geq 2$ ) through Eq. (10).

In Fig. 4 the rigid piston ( $\ell = 0$ ), the simply supported radiator ( $\ell = 1$ ) and the first two clamped radiators ( $\ell = 2, 3$ ) are considered ( $|p(\underline{r})|$ , normalized as a function of  $r/a$ ).



**Fig. 4:** Normalized  $|p|$  vs.  $r/a$  for the rigid piston ( $\ell = 0$ ) (solid curve), the simply supported radiator ( $\ell = 1$ ) (dotted curve) and the first two clamped radiators ( $\ell = 2, 3$ ) (dash-dotted and dashed curves, respectively) using Eq. (13). Here  $a/\lambda = 4$  and  $ka = 8\pi$ . The normalization is equal to  $(\ell + 1)/2 \rho_0 c V_s (ka)^2$ . The factor  $\ell + 1$  allows an easier comparison of the four curves.

The next comments concern the behavior of the terms  $\gamma_n$  in Eq. (14). From Eqs. (14)–(15) it follows that

$$r_- \approx \frac{1}{2}(a-r) \approx \frac{1}{2}a \approx \frac{1}{2}(a+r) \approx r_+, \quad r \ll a. \quad (17)$$

Therefore, when  $ka$  is large and  $r \rightarrow 0$  (with  $n$  not large), it follows from the results in Ref. [12], Sec. 10.1 that

$$|\gamma_n(k, r)| \approx \frac{|\cos \frac{1}{2}k(a-r)|}{\frac{1}{4}k^2 a^2}, \quad (18)$$

confirming the presence of zeros and the largely  $n$ -independent envelope of the curves in Fig. 3 near  $r = 0$ .

Finally, when  $r \gg a$  it follows from Eqs. (14)–(15) that

$$r_- \approx \frac{a^2}{4r}, \quad r_+ \approx r. \quad (19)$$

Therefore, from Ref. [12], Sec. 10.1,

$$\gamma_n(k, r) \approx \frac{\left(\frac{-ika^2}{4r}\right)^n}{1 \cdot 3 \cdot \dots \cdot (2n+1)} \frac{e^{-ikr_+}}{-ikr_+} \quad (20)$$

which shows an  $O(1/r^{n+1})$ -behavior of  $\gamma_n(k, r)$  as  $n \rightarrow \infty$ .

#### 4.2. Far-field expression

Using the Zernike expansion Eq. (3) of  $v(\sigma)$  it is shown in [1][Appendix B] that the following far-field approximation holds: when  $\underline{r} = (r \sin \theta, 0, r \cos \theta)$  and  $r \rightarrow \infty$ ,

$$p(\underline{r}) \approx i\rho_0 c k V_s \frac{e^{-ikr}}{r} a^2 \sum_{n=0}^{\infty} u_n (-1)^n \frac{J_{2n+1}(ka \sin \theta)}{ka \sin \theta}. \quad (21)$$

In the case of a rigid piston, it follows that

$$p(\underline{r}) \approx i\rho_0 c k a^2 V_s \frac{e^{-ikr}}{r} \frac{J_1(ka \sin \theta)}{ka \sin \theta}. \quad (22)$$

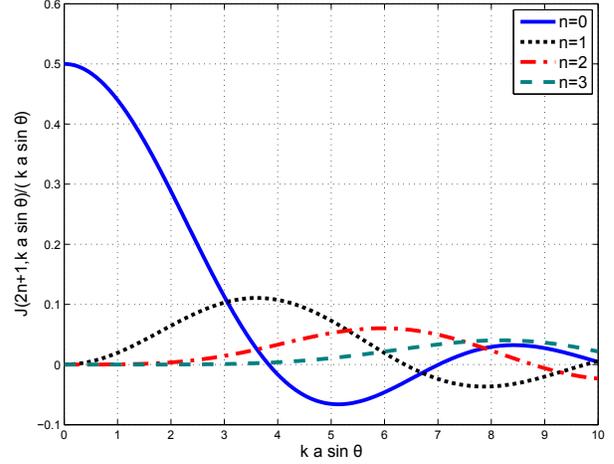
This is the familiar result for the far-field pressure of a rigid piston as can be found in the textbooks, see, e.g., Kinsler et al. [8, (8.35)]. In Fig. 5 a plot can be found of  $\left| \frac{J_{2n+1}(ka \sin \theta)}{ka \sin \theta} \right|$ ,  $n = 0, 1, 2, 3$ , as a function of  $ka \sin \theta$ .

For the simply supported radiator, case  $\ell = 1$  in Eq. (6), and for the clamped radiators, cases  $\ell \geq 2$  in Eq. (6), there is the far-field result of Stenzel [13],

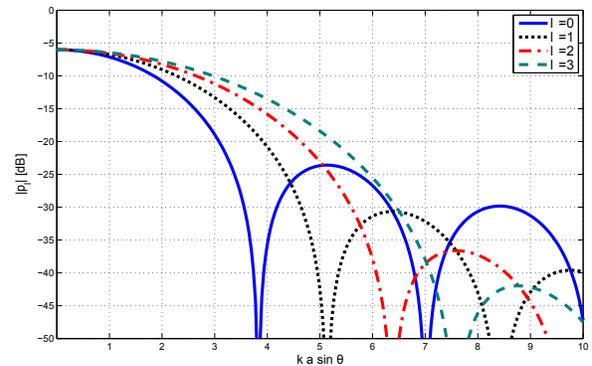
$$p(\underline{r}) \approx i\rho_0 c V_s k a^2 (\ell + 1)! 2^\ell \frac{e^{-ikr}}{r} \frac{J_{\ell+1}(ka \sin \theta)}{(ka \sin \theta)^{\ell+1}}, \quad (23)$$

also see Ref. [7, Sec. II]. Alternatively, the coefficients  $u$  in the Zernike expansion of the  $v^{(\ell)}$  in Eq. (6) are available per Eq. (10), and this gives the far-field approximation of  $p(\underline{r})$  via Eq. (21) in a different form. Either method yields the result shown in Fig. 6.

Some comments on the behavior of the terms  $J_{2n+1}(z)/z$ ,  $z = ka \sin \theta$ , as they occur in the series



**Fig. 5:**  $\left| \frac{J_{2n+1}(ka \sin \theta)}{ka \sin \theta} \right|$  vs.  $ka \sin \theta$ .



**Fig. 6:** Normalized  $|p|$  vs.  $ka \sin \theta$ , using the Zernike expansion of the  $v^{(\ell)}$  and Eq. (21) or (23).

in Eq. (21) are presented now. From the asymptotics of the Bessel functions, as given in Ref. [12, Eq. 9.3.1], it is seen that in the series in Eq. (21) only those terms contribute significantly for which  $2n + 1 \leq \frac{1}{2}e ka \sin \theta$ . In particular, when  $\theta = 0$ , it is only the term with  $n = 0$  that is non-vanishing, and this yields

$$p((0, 0, r)) \approx \frac{1}{2}i\rho_0cV_ska^2\frac{e^{-ikr}}{r}, \quad r \rightarrow \infty. \quad (24)$$

This is in agreement with what is found from Eq. (13) when only the term with  $n = 0$  is retained and  $r_+$  is replaced by  $r$ ,  $r_-$  is replaced by 0. For small values of  $ka$  the terms in the series Eq. (21) decay very rapidly with  $n$ . For large values of  $ka$ , however, a significant number of terms may contribute, especially for angles  $\theta$  far from 0.

## 5. POWER OUTPUT AND DIRECTIVITY

The power is defined as the intensity  $pv^*$  integrated over the plane  $z = 0$ . Thus, because  $v$  vanishes outside  $S$ ,

$$P = \int_S p(\sigma)v^*(\sigma)dS, \quad (25)$$

where  $p(\sigma) = p((\sigma \cos \psi, \sigma \sin \psi, 0))$  is the pressure at an arbitrary point on  $S$ .

In the notations of Fig. 1, King's result [15] reads

$$p(\underline{r}) = \frac{i\rho_0ck}{2\pi} \int_S v(\sigma) \frac{e^{-ikr'}}{r'} dS = i\rho_0ck \int_0^\infty \frac{e^{-z(u^2-k^2)^{1/2}}}{(u^2-k^2)^{1/2}} J_0(wu) V(u) u du,$$

where

$$(u^2-k^2)^{1/2} = \begin{cases} +i\sqrt{k^2-u^2}, & 0 \leq u \leq k \\ +\sqrt{u^2-k^2}, & k \leq u < \infty \end{cases}, \quad (26)$$

with  $\sqrt{\phantom{x}}$  non-negative, and

$$V(u) = \int_0^a J_0(u\sigma)v(\sigma)\sigma d\sigma, \quad u \geq 0, \quad (27)$$

is the Hankel transform (of order 0) of  $v$ . In [2] it is shown that

$$P = 2\pi i\rho_0ck \int_0^\infty \frac{V(u)V^*(u)}{(u^2-k^2)^{1/2}} u du. \quad (28)$$

This gives rise, via Eqs. (3) and (12), to the integrals

$$\int_0^\infty \frac{J_{2n+1}(au)J_{2n+1}(au)}{(u^2-k^2)^{1/2}} du. \quad (29)$$

and these have been evaluated in the form of a power series in  $ka$  in [2][Sec.V.B].

We next consider the directivity. With the usual approximation arguments in the Rayleigh integral representation of  $p$  in Eq. (1) there follows ( $\underline{r} = (r \cos \psi \sin \theta, r \sin \psi \sin \theta, r \cos \theta)$ )

$$p(\underline{r}) \approx i\rho_0ck \frac{e^{-ikr}}{r} V(k \sin \theta). \quad (30)$$

From this there results the directivity

$$D = \frac{4\pi|V(0)|^2}{\int_0^{2\pi} \int_0^{\pi/2} |V(k \sin \theta)|^2 \sin \theta d\psi d\theta} = \frac{2|V(0)|^2}{\int_0^{\pi/2} |V(k \sin \theta)|^2 \sin \theta d\theta}, \quad (31)$$

see Kinsler et al. [8], Sec.8.9. By Eqs. (2) and (27) it holds that  $V(0) = \frac{1}{2}a^2V_s$ , and from Eqs. (26) and (28) it can be shown that

$$\int_0^{\pi/2} |V(k \sin \theta)|^2 \sin \theta d\theta = \frac{1}{2\pi\rho_0ck^2} \Re[P]. \quad (32)$$

Consider the case that  $ka \rightarrow 0$ . In [2] it is derived that  $\Re[P] \approx \frac{1}{2}\pi\rho_0cV_s^2a^2(ka)^2$ . Therefore, as  $ka \rightarrow 0$

$$D \approx \frac{2(\frac{1}{2}a^2V_s)^2}{\frac{1}{2\pi\rho_0ck^2} \frac{1}{2}\pi\rho_0cV_s^2a^2(ka)^2} = 2, \quad (33)$$

or 3 dB; this limiting value 2 of  $D$  is also found in the case of a rigid piston [8] or a hemispherical source on an infinite baffle.

Next consider the case that  $ka \rightarrow \infty$ . It is shown in [2][Sec. V.D] that

$$D \approx \frac{2(\frac{1}{2}a^2V_s)^2}{\frac{1}{2\pi\rho_0ck^2} 2\pi\rho_0ca^2 \int_0^1 |v(a\rho)|^2 \rho d\rho} = \frac{\frac{1}{2}(ka)^2V_s^2}{\int_0^1 |v(a\rho)|^2 \rho d\rho} = C_v(ka)^2, \quad (34)$$

in which  $C_v$  is the ratio of the square of the modulus of the average velocity and the average of the square

of the modulus of the velocity (averages over  $S$ ). In case that  $v = v^{(\ell)}$ , the last member of Eq. (34) is given by  $(2\ell + 1)(\ell + 1)^{-1}(ka)^2$ ; in Kinsler et al. [8], end of Subsec. 8.9. this result for the case  $\ell = 0$  is given.

## 6. ESTIMATING VELOCITY PROFILES FROM ON-AXIS RADIATION DATA FOR FAR-FIELD LOUDSPEAKER ASSESSMENT

### 6.1. Estimating velocity profiles from on-axis radiation

The on-axis expression Eqs. (13)–(14) for the pressure can, in reverse direction, be used [1] to estimate the velocity profile on the disk from (measured) on-axis data via its expansion coefficients  $u_n$ . This can be effectuated by adopting a matching approach in which the coefficients  $u_n$  in the ‘theoretical’ expression Eqs. (13)–(14) are determined so as to optimize the match with measured data at  $M + 1$  points. Thus, one has for the pressure  $p_m = p((0, 0, r_m))$  due to the velocity profile  $v(\sigma) = V_s \sum_{n=0}^N u_n R_{2n}^0(\sigma/a)$  the expression

$$p_m = \frac{1}{2} \rho_0 c V_s (ka)^2 \cdot \sum_{n=0}^N (-1)^n j_n(kr_{m,-}) h_n^{(2)}(kr_{m,+}) u_n, \quad (35)$$

where  $r_m \geq 0$  and

$$r_{m,\pm} = \frac{1}{2} (\sqrt{r_m^2 + a^2} \pm r_m), \quad (36)$$

and  $m = 0, 1, \dots, M$ . With

$$A = (A_{mn})_{\substack{m=0,1,\dots,M, \\ n=0,1,\dots,N}}, \quad (37)$$

$$A_{mn} = \frac{1}{2} \rho_0 c V_s (ka)^2 j_n(kr_{m,-}) h_n^{(2)}(kr_{m,+}),$$

$$\underline{p} = [p_0, \dots, p_M]^T, \quad \underline{u} = [u_0, \dots, u_N]^T, \quad (38)$$

the relation between on-axis pressures  $p_m$  and coefficients  $u_n$  can be concisely written as

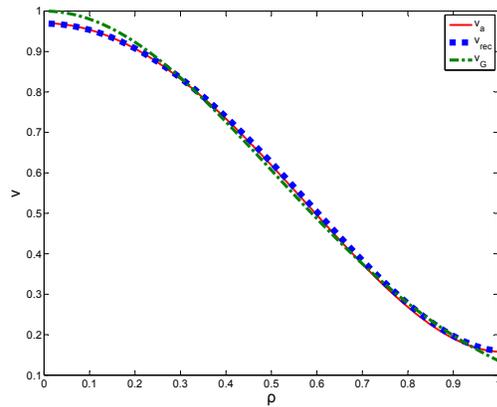
$$A \underline{u} = \underline{p}. \quad (39)$$

Now given a (noisy) on-axis data vector  $\underline{p}$  one can estimate the coefficients vector  $\underline{u}$  by adopting a least mean-squares approach for the error  $A \underline{u} - \underline{p}$ . This will be illustrated by a simulated experiment and, subsequently, by a real experiment, below. In the simulated experiment, we assume a loudspeaker with a

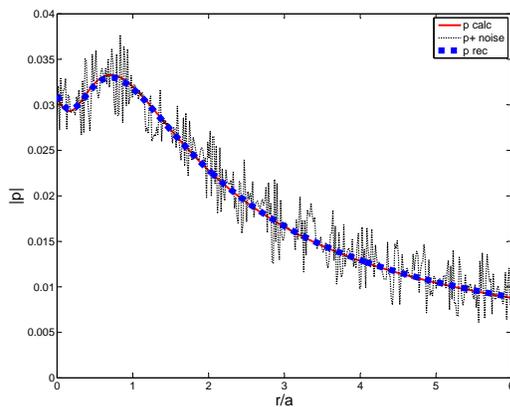
Gaussian velocity profile ( $\alpha = 2$ ), as shown in Fig. 7-a curve  $v_G$  (dash-dotted) and given by the left-hand side of Eq. (11). This profile is approximated using three Zernike coefficients ( $u_0 = 0.4323$ ,  $u_1 = -0.4060$ ,  $u_2 = 0.1316$ ) given by the right-hand side of Eq. (11), and this leads to the velocity profile  $v_a$  (solid curve) in Fig. 7-a. It can be seen from Fig. 7-a that including three Zernike terms provides a fair approximation ( $3 \cdot 10^{-2}$  absolute accuracy on the whole range). Using the three coefficients of the approximated velocity profile, the sound pressure was calculated by applying Eq. (13) and plotted in Fig. 7-b as  $p_{calc}$  (solid curve). Then random white noise was added to  $p_{calc}$  as shown as  $p + \text{noise}$  in Fig. 7-b (dotted curve). Subsequently, the inversion procedure was followed by using the noisy pressure data vector  $\underline{p}$  to estimate the coefficients vector  $\underline{u}$  by adopting a least mean-squares approach for the error  $A \underline{u} - \underline{p}$  (see Eq. (39)). Using the recovered three Zernike coefficients the velocity profile and pressure data were calculated and plotted in Fig. 7-a (thick dotted curve) and Fig. 7-b (thick dotted curve), respectively. It appears that the inversion procedure is rather robust against noise since the calculated and recovered pressure curves in Fig. 7-b are almost coincident.

For the second experiment we measured a loudspeaker (vifa MG10SD09-08,  $a = 3.2$  cm) in an IEC-baffle [17], at 10 near-field positions ( $r_m = 0.00, 0.01, 0.02, 0.03, 0.04, 0.05, 0.07, 0.10, 0.13, 0.19$  m), and finally the bottom curve for the far-field at 1 m distance. The SPL is plotted in Fig. 8. Figure 8 clearly shows that the near field differs from the far field, in particular at higher frequencies. The lower curve is somewhat noisy because the amplification of the microphone amplifier was kept the same for all measurements.

For a particular frequency at 13.72 kHz ( $ka = 8.0423$ ), the magnitude of the sound pressure is plotted in Fig. 9 (solid curve ‘ $p$  meas’). Using the same procedure as described above for the first simulation, the inverse process was followed by using the ten measured near-field pressure data points to estimate the coefficients vector  $\underline{u}$ . Using four Zernike coefficients the pressure data were recovered and plotted in Fig. 9 (dotted curve ‘ $p$  rec’). It appears that the two curves show good resemblance to each other and that only four coefficients are needed to provide a very good description of the near-field at

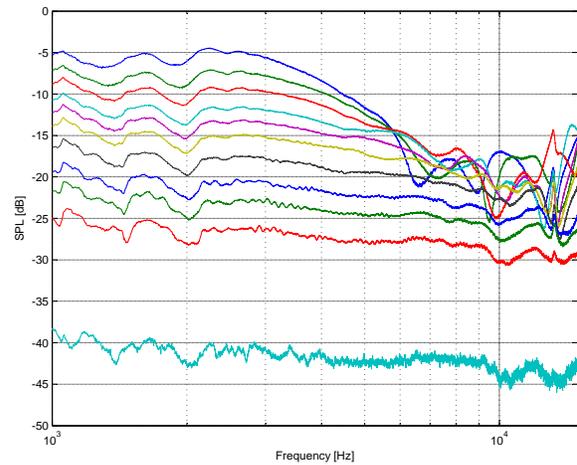


(a)



(b)

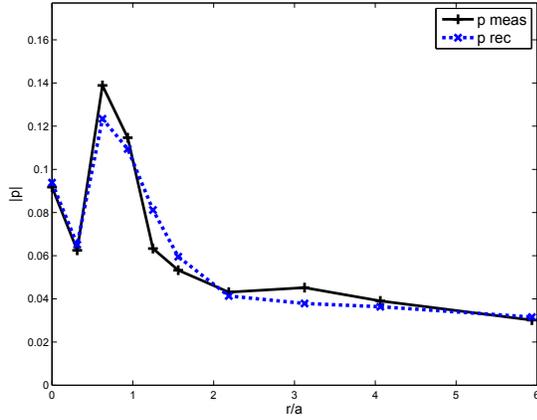
**Fig. 7:** Simulated experiment. (a) Gaussian velocity profile ( $\alpha = 2$ )  $v_G$  vs.  $\rho$  (dash-dotted curve, given by the LHS of Eq. (11)). Approximated velocity profile  $v_a$  using the series at the RHS of Eq. (11), truncated at  $n = 2$  (solid curve). From noisy pressure data recovered velocity profile  $v_{rec}$  (thick dotted curve). (b) Sound pressure using Eq. (13) and  $ka = 8$  ( $p_{calc}$ , solid curve). Pressure with added noise ( $p+noise$ , dotted curve). Recovered pressure data ( $p_{rec}$  thick dotted curve).



**Fig. 8:** Measured SPL for the loudspeaker (vifa MG10SD09-08,  $a = 3.2$  cm) in an IEC-baffle [17], at 10 near-field positions from upper curve (0.00, 0.01, 0.02, 0.03, 0.04, 0.05, 0.07, 0.10, 0.13, 0.19 m), and finally the bottom curve for the far-field at 1 m distance.

rather high frequencies (13.72 kHz). Furthermore, it appears that using these four coefficients, the calculated sound pressure level at 1-m distance yields -42 dB. The measured value at that far-field point is -44 dB. These values match rather closely, even though the cone vibrates not fully circularly symmetric anymore at the used frequency of 13.72 kHz, due to break-up behavior. This match provides a proof of principle as the far-field measurement point was not used to determine the Zernike coefficients.

In the experiments just described, no particular effort was spent in forming and handling the linear systems so as to have small condition numbers. The condition numbers, the ratio of the largest and smallest non-zero singular value of the matrix  $A$  in Eq. (37), equals 50 in the case of the loudspeaker experiment leading to Fig. 9. In practical cases the number of required Zernike coefficients will be less than, say, six. This will not cause numerical difficulties. Furthermore, such a modest number of coefficients already parameterizes a large set of velocity profiles.



**Fig. 9:** Measured loudspeaker at 13.72 kHz ( $ka = 8.0423$ ,  $p_{\text{meas}}$ , solid curve) vs.  $r/a$ . Recovered pressure data ( $p_{\text{rec}}$  dotted curve).

## 6.2. Far-field assessment from on-axis measurements

In Keele [9] a method is described to assess low-frequency loudspeaker performance in the on-axis far-field from an on-axis near-field measurement. In the case of the rigid piston, the on-axis pressure  $p(r) = p((0, 0, r))$  is given by Eq. (16). Now assume  $ka \ll 1$ . When  $r \ll a$  it holds that

$$\sin\left(\frac{1}{2}k\sqrt{r^2 + a^2} - r\right) \approx \sin\left(\frac{1}{2}ka\right) \approx \frac{1}{2}ka, \quad (40)$$

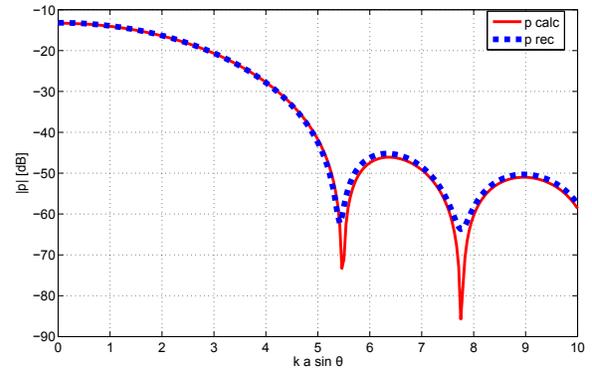
and, when  $r \gg a$  it holds that

$$\sin\left(\frac{1}{2}k(\sqrt{r^2 + a^2} - r)\right) \approx \sin\left(\frac{ka^2}{4r}\right) \approx \frac{ka^2}{4r}. \quad (41)$$

Therefore, the ratio of the moduli of near-field and far-field on-axis pressure is given by  $2r/a$ . This is the basis of Keele's method; it allows far-field loudspeaker assessment without having to use an anechoic room.

With the inversion procedure to estimate velocity profiles from on-axis data (which are taken in the relative near-field) as described in Sec. 6.1 together with the forward calculation scheme for the far-field as described in Sec. 4.2, it is now possible to generalize Keele's scheme. This is illustrated by comparing the far-field responses pertaining to the two

sets of Zernike coefficients occurring in the Gaussian simulated experiment, see Fig. 7 in Sec. 6.1. Using Eq. (21) the normalized far-field pressure is plotted in Fig. 10 as  $p_{\text{calc}}$  (solid curve) and  $p_{\text{rec}}$  (dotted) curve, respectively ( $\alpha = 2, ka = 8$ ), where the normalization is such that the factor in front of the series at the right-hand side of Eq. (21) equals unity. It appears that the two curves are very similar. This confirms that the  $u_n$  obtained from the noisy near-field measured pressure data yield a good estimate of the far-field spatial pressure response.



**Fig. 10:** Simulated experiment Gaussian radiator ( $\alpha = 2$ ). Normalized sound pressure in the far-field using Eq. (21) and  $ka = 8$  ( $p_{\text{calc}}$ , solid curve). Recovered normalized far-field pressure data ( $p_{\text{rec}}$  dotted curve).

## 7. DISCUSSION AND OUTLOOK

This paper has considered a method to perform forward and inverse sound pressure computations for circular radiators with a non-uniform velocity profile. In the forward problem, the velocity profile is assumed to be known and the on-axis and far-field sound pressure are expressed analytically in terms of Zernike expansion coefficients of the velocity profile and (spherical) Bessel (and Hankel) functions. In the inverse problem, the velocity profile is unknown and is estimated in terms of Zernike expansion coefficients from on-axis pressure data by adopting a matching approach based on the analytic result for the on-axis pressure. Well-behaved velocity profiles are already adequately represented by only a few terms of their Zernike expansion. Therefore, the Zernike series approach is more convenient for

both the forward problems and the inverse problem than, for instance, an approach based on expansions involving the family of rigid, simply supported and clamped radiators. The forward and inverse method is proposed for use in assessment of the far-field of a loudspeaker without the need for an anechoic room. Here, the Zernike coefficients of the velocity profile are estimated from the on-axis (relatively near-field) data, and these coefficients are used in the forward scheme to compute the far-field. This assessment procedure has not been fully worked out in the present paper due to a variety of practical issues that need to be addressed. Among these practical issues are

- choice of the on-axis measurement points,
- condition of the linear systems that arise,
- influence of  $ka$ ,
- influence of noise,
- influence of misalignment of the measurement points,
- influence of inclination of the measurement axis,
- incorrect setting of the radius of the radiator,

while various combinations of these issues should also be considered. The authors intend to work out the method for the loudspeaker assessment with attention for the above mentioned points.

In this paper, the theory has been developed for flat radiators. However, the basic result for the on-axis pressure as a series expansion, in terms of Zernike coefficients and spherical Bessel and Hankel functions, has been generalized to the case of dome-shaped radiators [1]. It is, therefore, to be expected that both the forward and the inverse methods can also be generalized to the case of dome-shaped radiators. The authors intend to develop the method both on a theoretical and practical level for dome-shaped radiators as well.

## 8. CONCLUSIONS

Zernike polynomials are an efficient and robust method to describe velocity profiles of resilient sound radiators. A wide variety of velocity profiles, including the rigid piston, the simply supported radiator,

the clamped radiators, Gaussian radiators as well as real loudspeaker drivers, can be approximated accurately using only a few terms of their Zernike expansions. This method enables one to solve both the forward as well as the inverse problem. With the forward method the on-axis and far-field off-axis sound pressure are calculated for a given velocity profile. With the inverse method the actual velocity profile of the radiator is calculated using (measured) on-axis sound pressure data. This computed velocity profile allows the extrapolation to far-field loudspeaker pressure data, including off-axis behavior, without the use of anechoic rooms.

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