Symbolic analysis of loudspeaker cross over filters

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Keywords: RLC, loudspeaker cross over filter, transfer function

Abstract

Computer algebra can be used to compute the symbolic transfer function of a loudspeaker cross over filter as a function of frequency, filter components and loudspeaker impedance. We describe a key ingredient of this kind in a program to analyse RLC-loudspeaker cross-over filters developed and implemented at the Philips Research Lab., Eindhoven.

1. Introduction

In this paper we study linear networks consisting of one-ports only, which we call the branches. For a branch two variables are of primary interest: the voltage across the branch and the current through the branch. For example, a resistance R is characterised by $V = IR$, and a capacitance C is characterised by $i = Cdv/dt$, or $I = CsV$ in the frequency domain.

When a number of branches are connected, we have a lumped network. An example of a loudspeaker modelled by a lumped network is pictured in Figure 1.

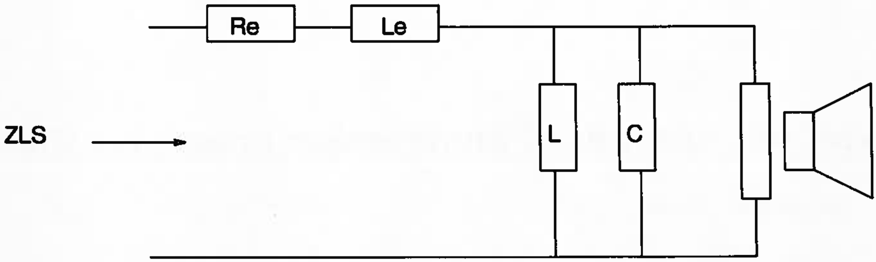

Figure 1. A lumped element model of a loudspeaker

In Figure 1 the symbols have the following meaning:

Re: DC-resistance of the voice coil

Le: Self inductance of the voice coil

L: Electric analogue of the cone mass

C: Electric analogue of the spider spring

Any lumped network obeys the Kirchhoff laws giving linear constraints on the branch currents and voltages. Network topology (also called configuration) deals with those properties of the lumped networks which are related to the interconnection of the branches only.

In filter design, one usually starts by proposing a reasonable network topology, and by selecting initial element values. The actual frequency response is then calculated and compared with the specified response. In this approach, the response of a network has to be calculated at many frequency points. Obviously, if a symbolic transfer function, that is, a symbolic expression for V_{out}/V_{in} (or I_{out}/I_{in}, etc.), can be computed first, then repeated analysis of the network follows by substitution of the numerical values.

2. Acknowledgements

This paper is based on a program developed and implemented by Ing. R.M. Aarts and R.G.J Janmaat at the Philips Research Laboratories, Eindhoven (see [2]).

3. The Problem

For some years a filter optimisation package FOPT, developed at the Philips Research Laboratories, is successfully used (see [1]). The inputs for FOPT are loudspeaker data (e.g., loudspeaker characteristics), target data (e.g., desired sound pressure response and power response) and a PASCAL procedure RESP which is used to compute the input impedance and the voltage transfer function of the filter as a function of frequency f, filter components C and the loudspeaker impedance LS.

The numerical analysis of the filter is done by the user, i.e., the user must write the PASCAL code based on the topology of the filter. The symbolical analysis of the filter is also performed by the user and results in the Laplace domain coefficients of the voltage transfer function as a function of filter components C and the lumped element parameters L of the s-domain model used to represent the loudspeaker.

The drawbacks of this program are clear: user intervention is required for both the numerical and the symbolical analysis on a case by case basis. The (PASCAL) code to compute the numerical transfer function can be computed from the symbolic transfer function of the RLC-filter. Therefore, the main task is to use computer algebra to compute the symbolic transfer function of a filter.

4. The Mathematical Model

We will restrict ourselves to linear time-invariant two-port networks that consist of resistor R, conductor G, inductor L and capacitor C elements only. Figure 2 shows a general two-port network. It is driven by an ideal unity current source I_γ and loaded with a complex impedance Z_{LS}. The network consists of one-ports only. The nodes n_i are numbered from 0 to N. The input nodes are n_0 and n_γ. The output nodes are n_α and n_β. They are referred to as I/O-nodes. Figure 3 shows a simple example. There, for instance, Y_2 and Y_5 are one-ports connecting the nodes n_0 and n_2. The quantity Y_j also stands for the admittance of the one-port, which is the inverse of its impedance.

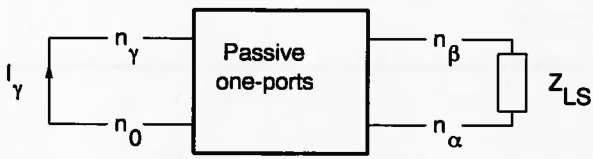

Figure 2. A two-port network

For the network of Figure 2 with node n_0 grounded and a unity current source, the input impedance is defined as

$$Z_\gamma(s,x) = V_\gamma(s,x), \tag{4.1}$$

where s is the frequency variable in the Laplace domain, x is the parameter vector containing load and network parameters and V_γ is the input node voltage with respect to the ground. The voltage transfer function for the network of Figure 2 is given by

$$H_{\beta\alpha} = \frac{V_\beta(s,x) - V_\alpha(s,x)}{V_\gamma(s,x)}, \tag{4.2}$$

where n_0, n_γ, n_α and n_β belong to $\{0, 1, \ldots, N\}$ such that n_0, n_γ are input and n_α, n_β are output nodes and V_i is the node voltage with respect to the ground.

The indefinite nodal method (see [4]) provides a set of non-singular linear equations from which all node voltages can be solved. Let $Y = (y_{ij})$ be the matrix whose

(i) diagonal entries are $y_{jj} = \sum$ admittances connected to n_j.

(ii) off-diagonal entries are $y_{jk} = -\sum$ admittances connected to n_j and n_k.

The indefinite nodal equation is then defined as follows

$$YV = J, \tag{4.3}$$

where $V = (V_0, \ldots, V_N)^\top$ denotes the node voltage vector and $J = (J_0, \ldots, J_N)$,

$$J_j = \sum \text{ currents from independent sources entering } n_j.$$

To transform the indefinite nodal equation into a linear nonsingular system that can be solved, we shall use the fact that grounding the k^{th} node comes down to deleting the k^{th} row and column of Y.

A two-port network consisting of one-ports only has a symmetric admittance matrix ($Y = Y^\top$) and an admittance matrix of a one-port connected between nodes k and l has entries $y_{kk} = y_{ll} = y$ and $y_{kl} = y_{lk} = -y$, where y denotes the admittance connected to k.

4.1. *A loudspeaker*

A loudspeaker can be modelled by a lumped element network (see Figure 1). The impedance of a lumped element model is given by

$$Z_{LS} = Re + sLe + \frac{sRme}{s^2Q/\omega_0 + s + Q\omega_0}.$$

Given the symbolic transfer function $H(s)$ of the corresponding filter, we make the following transformation

$$s \mapsto \frac{2}{T}\frac{z-1}{z+1},$$

where T is the sampling frequency. Once the z-domain transfer function is known, it can be made suitable for implementation on a Digital Signal Processor (see [1]).

5. The Mathematics

To solve the symbolic linear system (4.3) with the n_0 node grounded, we shall use computer algebra. For a small number of nodes this can be done directly. If the number of nodes becomes large, it is worthwhile to reduce the number of nodes.

To minimise the number of node voltages and network parameters, we shall use the following reduction rules:

- *The P-reduction rule.* In a two-port network M parallel one-port connections can be replaced by one new one-port with new admittance $y_{M'}$ given by the sum of the old admittances

$$y_{M'} = \sum_{k=1}^{M} y_k. \tag{5.1}$$

- *The S_M-reduction rule.* An internal node voltage can be removed from the nodal equation by a star to maze transformation. Here, a star shape with $M+1$ nodes and M branches is transformed into a maze structure with M nodes and $M(M-1)/2$ branches. Thus, the number of network branches increases by $M(M-3)/2$, but, more significantly, the number of node voltages reduces by one. If the center node is n_l, then the new admittances are given by

$$y_{ij} = \frac{y_{il}y_{lj}}{\sum_{k=1}^{M} y_{kl}}. \tag{5.2}$$

A star to maze transformation can introduce new parallel one-ports. Remark that, if the reduction of a star structure with $M \geq 2$ branches into a maze structure is followed by a parallel reduction, then the resulting structure has precisely M new star shaped structures with at least $M-1$ branches. This implies that a $P \circ S_M$ reduction should be followed by a $P \circ S_{M-1}$ reduction if $M > 2$ and by $P \circ S_M$ reduction if $M = 2$. This yields the following reduction algorithm that eliminates all internal nodes of a star structure with M branches.

```
apply P reduction rule
while `there are internal node voltages` do
   apply P*S_M reduction rule;
   if M >2 then
      M := M-1;
   else
      M := M+1;
od
```

After the algorithm has been applied to all star structures, we can ground the n_0 node and the definite nodal equation can be solved using Cramer's rule. This yields the symbolic voltage transfer function $H_{\beta\alpha}$ for the network.

Next we illustrate the algorithm with a simple example. Consider a low pass loudspeaker crossover filter, where all one-ports are represented as admittances.

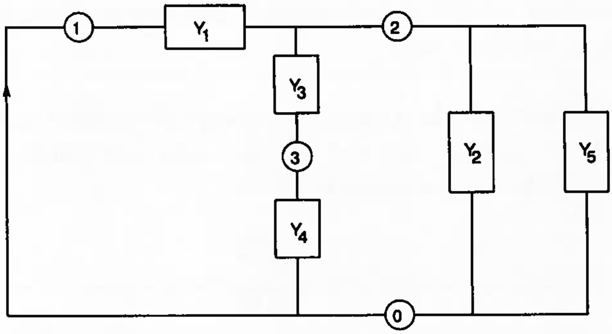

Figure 3. Example of a network

Set

$$\begin{aligned} Y_1 &= 1/(sL_1 + R_{L_1}), \\ Y_2 &= sC_1, \\ Y_3 &= sC_2, \\ Y_4 &= 1/R_1, \\ Y_5 &= 1/Z_{LS}. \end{aligned}$$

Using the definitions we find $\gamma = 1$, $\alpha = 0$, $\beta = 2$ and

$$x = (L_1, R_{L_1}, C_1, C_2, R_1, Z_{LS})^{\mathsf{T}}.$$

The indefinite nodal equation is given by

$$\begin{pmatrix} Y_2 + Y_4 + Y_5 & 0 & -Y_2 - Y_5 & -Y_4 \\ 0 & Y_1 & -Y_1 & 0 \\ -Y_2 - Y_5 & -Y_1 & Y_1 + Y_2 + Y_3 + Y_5 & -Y_3 \\ -Y_4 & 0 & -Y_3 & Y_3 + Y_4 \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Minimisation of the indefinite nodal equation using the reduction algorithm proceeds as follows. First the P-cycle will detect a parallel port. A new admittance element is created, say Y_6, and the nodal equation becomes

$$\begin{pmatrix} Y_4 + Y_6 & 0 & -Y_6 & -Y_4 \\ 0 & Y_1 & -Y_1 & 0 \\ -Y_6 & -Y_1 & Y_1 + Y_3 + Y_6 & -Y_3 \\ -Y_4 & 0 & -Y_3 & Y_3 + Y_4 \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Next the $P * S_2$ step will detect the internal node voltages V_3 as candidate for minimisation and the nodal equation becomes

$$\begin{pmatrix} Y_8 & 0 & -Y_8 \\ 0 & Y_1 & -Y_1 \\ -Y_8 & -Y_1 & Y_1 + Y_8 \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},$$

where

$$\begin{aligned} Y_6 &= Y_2 + Y_5, \\ Y_7 &= Y_3 Y_4/(Y_3 + Y_4), \\ Y_8 &= Y_6 + Y_7. \end{aligned}$$

There are no internal nodes nor parallel structures left and the algorithm has come to a stop. To solve the resulting nodal equation, we ground n_0 and then solve the remaining system using Cramer's rule.

The transfer functions of the filter are

$$\begin{aligned} Z_1 = V_1 &= \frac{Y_1 + Y_8}{Y_1 Y_8}, \\ H_{20} = V_2/V_1 &= \frac{Y_1}{Y_1 + Y_8}. \end{aligned}$$

Substituting the values of Y_i and the numerical value of x,

$$x = (1, 1, 1, 1, 1, Z)^{\mathsf{T}},$$

we obtain the transfer function as a function of s and the impedance $Z_{LS} = Z$:

$$H_{20}(s, Z) = \frac{Z}{1 + Z + (1 + 2Z)s + Zs^2}.$$

6. The Computer Algebra

The above described method to compute the symbolic transfer function of linear time-invariant two-port networks is implemented using the REDUCE computer algebra system. The program is called "netprogram" and has two main functions:

- from the filter specification, the program generates PASCAL code that can be used to calculate the input impedance and symbolic s-domain voltage transfer function of the filter;
- from the filter component data, lumped elements parameters of the loudspeaker and the symbolic s-domain voltage transfer function, the program generates the z-domain biquad coefficients that can be used as a start value for the curve-fitting optimisation package CURFIT.

The program listings are presented in [2].

7. Interpretation of the Results

In filter optimisation, a network topology is first proposed, and initial element values are selected. The actual frequency response is then calculated and compared with the specified response. In this approach, the response of a filter has to be calculated at many frequency points. Using computer algebra one can solve the symbolic nodal equation. For a small number of nodes this can be done directly. For a large number of nodes one can use the reduction rules as proposed in §5. As a result we find a symbolic transfer function V_{out}/V_{in} and the electrical input impedance Z_i. Given these symbolic transfer functions, repeated analysis of the filter follows by substitution of the numerical values.

8. Conclusion and Discussion

The use of computer algebra enables us to write a program that computes the symbolic transfer function of a filter based on the topology of the filter. Furthermore, given the symbolic transfer function, the numerical transfer function follows by substitution. The computer algebra program can be used to generate the (PASCAL) code which can be used to calculate the voltage transfer function and input impedance of a filter directly (see [2]).

9. References

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